Alternate derivation of Padmanabhan's differential bulk-surface relation in General Relativity

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March 5, 2013

Abstract

A differential bulk-surface relation of the lagrangian of General Relativity has been derived by Padmanabhan. This has relevance to gravitational information and degrees of freedom. An alternate derivation is given based on the differential form gauge theory formulation of gravity due to Göckeler and Schücker. Also an entropy functional of Padmanabhan and Paranjape can be rewritten as the Göckeler and Schücker lagrangian.

PACS codes: 04.50.Kd, 04.20.Cv, 04.20.Ha, 05.90.+m

1 Introduction

Padmanabhan, in the context of standard General Relativity(GR), has derived a differential relation between the bulk and surface terms of the Einstein-Hilbert(EH) lagrangian. He has done this in 3 different ways. [1], [2], [3], [4]. This splitting of the GR lagrangian was known historically, but it was Padmanabhan who found a relation between them. This implies that the bulk and surface terms, one being the derivative of the other, contain the same information. According to Padmanabhan this has relevance to both the physics of black holes and the Universe as a whole. In fact Padmanabhan claims the gravitational degrees of freedom of a region are on its boundary.

Padmanabhan's general procedure to study relativity is to use accelerated noninertial Rindler frames the same way Einstein used inertial frames. It will not be possible to give a detailed exposition of all of Padmanabhan's results in this short paper but see [1] or [2] for a synopsis.

1.1 The Göckeler and Schücker formulation

The mathematics of this paper is based on the differential form gauge theory formulation of GR of Göckeler and Schücker(G&S). [7] The G&S action for gravity is

$$S_{GS} = -\frac{1}{32\pi G}\int_{\mathcal{U}} R^{\alpha}{}_{\beta} \wedge \star(f^{\beta} \wedge f_{\alpha}) \tag{1}$$

where G is Newtons gravitational constant, $\mathcal{U}$ is some open set on a manifold or fiber bundle taken to be 4-dimensional spacetime with $\alpha, \beta = 0, \ldots, 3$. The fs are an oriented basis of the cotangent space,

$f^\alpha = f^\alpha_{\ \beta}\, dx^\beta$, $\mathbb{R}^4$ valued 1-forms. $*$ is the Hodge star, the duality (complementary) map amongst basis forms.

$$R = d\omega + \omega \wedge \omega \qquad R^{\alpha\beta} = d\omega^{\alpha\beta} + \omega^\alpha_{\ \gamma} \wedge \omega^{\gamma\beta} \tag{2}$$

where ω is a $\mathbf{gl_4}$ valued 1-form. d is the exterior derivative. and D is the exterior covariant derivative. Thus R is also a $\mathbf{gl_4}$ valued 2-form. (Note: Symbolically $R = D\omega = d\omega + \omega \wedge \omega$. However ω does not transform in the same linear representation of the gauge group as defined by D. So $R \neq D\omega$) Indices are raised and lowered with the metric $g_{\alpha\beta}$ whose signature here is $+ - --$. In Mathematics ω is called the connection and in Physics it is called the potential. It is the generator of parallel translations and is the gauge field. [7] In General Relativity the torsion $T = 0$. In this G&S formulation $0 \neq T = Df = df + \omega \wedge f$ and is an $\mathbb{R}^4$ valued 2-form. According to Wald torsion is necessary for spinorial matter. [5] Also according to Aldrovandi, Barros, and Pereira [6] non0 torsion is incompatible with the Principle of Equivalence because then objects do not move on geodesics. It may be mentioned that Göckeler and Schücker clarify the gauge structure of GR and the difference between diffeomorphisms and coördinate transformations, often confused in the literature. [7, pp 75, 87]

The G&S Action, Eqn (1), is a gauge theory of GR and reduces to the EH form under contraction of certain indices. This can be seen as follows.

GR requires a holonomic frame, namely $f^\alpha = dx^\alpha$. Otherwise in a nonholonomic frame $f^\alpha = f^\alpha_{\ \zeta} dx^\zeta$ where the $f^\alpha_{\ \zeta} \,\epsilon\, \mathbb{R}^4$, $\zeta = 0, \cdots, 3$. Using the definition of $*$, $*(f^\beta \wedge f_\alpha) = *(f^\beta \wedge f^\zeta g_{\zeta\alpha}) \to$
$*(g_{\zeta\alpha} dx^\beta \wedge dx^\zeta) = \varepsilon_{\zeta\eta\gamma\delta} g_{\iota\alpha} g^{\iota\zeta} g^{\eta\beta} dx^\gamma \wedge dx^\delta \, |\det g_{\iota\kappa}|^{\frac{1}{2}}$. ε is the totally antisymmetric ε-tensor or Levi-Civita tensor. Then $R^\alpha_{\ \beta} \wedge \star (g_{\zeta\alpha} f^\beta \wedge f^\zeta) \to -R^\alpha_{\ \beta} \wedge \varepsilon_{\zeta\eta\gamma\delta} g_{\alpha\iota} g^{\iota\zeta} g^{\eta\beta} (dx^\gamma \wedge dx^\delta) \sqrt{-g} = R^{\alpha\beta} \varepsilon_{\alpha\beta\gamma\delta} dx^\gamma \wedge dx^\delta \sqrt{-g}$ where $g = \det g_{\iota\kappa}$. Finally particular coördinates are chosen, i.e. the gauge is fixed.

Now the EH lagrangian requires a scalar curvature so $R^{\alpha\beta} := \frac{1}{2} R^{\alpha\beta}_{\ \ \ \gamma\delta} dx^\gamma \wedge dx^\delta$ is contracted to $R^{\alpha\beta} \to 2 \cdot \frac{1}{2} R^{\alpha\beta}_{\ \ \ \alpha\beta} dx^\alpha \wedge dx^\beta$. A factor of 2 is included because there are $2\times$ less indices. Substituting into Eqn. (1) gives

$$\begin{aligned} S_{GS}[\beta, \omega] &= -\frac{1}{32\pi G} \int_{\mathcal{U}} R \varepsilon_{\alpha\beta\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta \sqrt{-g} \\ &= \frac{1}{32\pi G} \int_{\mathcal{U}} \sqrt{-g} dx^4 R = S_{EH}[dx, \Gamma] \end{aligned} \tag{3}$$

where $dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta = 4! dx^{[\alpha} \otimes dx^\beta \otimes dx^\gamma \otimes dx^{\delta]}$ with the brackets denoting antisymmetrization. So $\varepsilon_{\alpha\beta\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta = \dfrac{1}{4!} 4! dx^{[\alpha} \otimes dx^\beta \otimes dx^\gamma \otimes dx^{\delta]} = dx^0 dx^1 dx^2 dx^3 = d^4 x$ assuming the latter is oriented. Γ is the notation for the connection ω in GR.

The variation f in $S_{GS}[\beta, \omega]$ to obtain the Einstein equations(EE) is done in G&S

$$G_{\alpha\beta} = 8\pi G \tau_{\alpha\beta} \tag{4}$$

where the Einstein tensor $G^\gamma_{\ \delta} = R^{\alpha\gamma}_{\ \ \ \alpha\delta} - \frac{1}{2} R^{\alpha\beta}_{\ \ \ \alpha\beta} \delta^\gamma_{\ \delta}$ and the energy-momentum tensor $\tau_{\alpha\beta}$ is defined as $\tau_\alpha = \frac{1}{6} \tau_\alpha^{\ \beta} \varepsilon_{\beta\gamma\delta\zeta} f^\gamma \wedge f^\delta \wedge f^\zeta$.

Varying ω Eqn. (1) leads to the torsion-spin equations

$$T^\alpha \wedge f^\beta \varepsilon_{\gamma\delta\alpha\beta} = -8\pi G S_{\gamma\delta} \tag{5}$$

where $S_{\gamma\delta}$ is the 3-form spin tensor.

1.2 The bulk and surface terms

Substituting $R^{\alpha}{}_{\beta} = d\omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\gamma} \wedge \omega^{\gamma}{}_{\beta}$ and using the Hodge star $*$ in the G&S action, Eqn (1), gives

$$\begin{aligned} S_{GS}[\beta,\omega] &= -\frac{1}{32\pi G}\left[\int_{\mathcal{U}} \omega^{\alpha}{}_{\kappa} \wedge \omega^{\kappa}{}_{\beta} \wedge \varepsilon_{\zeta\eta\gamma\delta} g_{\iota\alpha} g^{\iota\zeta} g^{\eta\beta} f^{\gamma} \wedge f^{\delta}\sqrt{-g}\right. \\ &\qquad \left. + \int_{\mathcal{U}} d\omega^{\alpha}{}_{\beta} \wedge \varepsilon_{\zeta\eta\gamma\delta} g_{\iota\alpha} g^{\iota\zeta} g^{\eta\beta} f^{\gamma} \wedge f^{\delta}\sqrt{-g}\right] \end{aligned}$$

Simplifying

$$\begin{aligned} S_{GS}[\beta,\omega] &= -\frac{1}{32\pi G}\left[\int_{\mathcal{U}} \omega^{\alpha}{}_{\kappa} \wedge \omega^{\kappa\beta} \wedge \varepsilon_{\alpha\beta\gamma\delta} f^{\gamma} \wedge f^{\delta}\sqrt{-g}\right. \\ &\qquad \left. + \int_{\mathcal{U}} d\omega^{\alpha\beta} \wedge \varepsilon_{\alpha\beta\gamma\delta} f^{\gamma} \wedge f^{\delta}\sqrt{-g}\right] \end{aligned} \tag{6}$$

The second term on the right hand side(rhs) can be made an exact differential in the following manner.

$$\begin{aligned} & d(\omega^{\alpha\beta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}) \\ =\ & d(\omega^{\alpha\beta}) \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} - \omega^{\alpha\beta} \wedge (df^{\gamma}) \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} \\ & \quad + \omega^{\alpha\beta} \wedge f^{\gamma} \wedge (df^{\delta})\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} - \omega^{\alpha\beta} \wedge f^{\gamma} \wedge f^{\delta} \wedge (d\sqrt{-g})\varepsilon_{\alpha\beta\gamma\delta} \end{aligned} \tag{7}$$

because in any single coördinates $\varepsilon_{\alpha\beta\gamma\delta}$ being constants, $d\varepsilon_{\alpha\beta\gamma\delta} = 0$ and ω and f are 1-forms. Note $-\omega^{\alpha\beta} \wedge (df^{\gamma}) \wedge f^{\delta}\varepsilon_{\alpha\beta\gamma\delta} = \omega^{\alpha\beta} \wedge f^{\gamma} \wedge (df^{\delta})\varepsilon_{\alpha\beta\gamma\delta}$. Substituting (7) in (6) and rearranging gives

$$\begin{aligned} S_{GS}[\beta,\omega] &= -\frac{1}{32\pi G}\left[\int_{\mathcal{U}} \omega^{\alpha}{}_{\zeta} \wedge \omega^{\zeta\beta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} - 2\int_{\mathcal{U}} \omega^{\alpha\beta} \wedge f^{\gamma} \wedge (df^{\delta})\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}\right. \\ &\qquad \left. + \omega^{\alpha\beta} \wedge f^{\gamma} \wedge f^{\delta} \wedge (d\sqrt{-g})\varepsilon_{\alpha\beta\gamma\delta} + \int_{\mathcal{U}} d(\omega^{\alpha\beta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta})\right] \end{aligned} \tag{8}$$

The last term on the rhs is an exact differential and consequently by Stokes Theorem, $\int_{\mathcal{U}} dA = \int_{\partial\mathcal{U}} A$ with $\partial\mathcal{U}$ the boundary of $\mathcal{U}$, is a boundary term. The bulk and the surface terms become respectively

$$\begin{aligned} -\frac{1}{32\pi G}\int_{\mathcal{U}} \Big\{ & \left[\omega^{\alpha}{}_{\zeta} \wedge \omega^{\zeta\beta} \wedge f^{\gamma} \wedge f^{d} - 2\omega^{\alpha\beta} \wedge f^{\gamma} \wedge (df^{\delta})\right]\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} \\ & + \omega^{\alpha\beta} \wedge f^{\gamma} \wedge f^{\delta} \wedge (d\sqrt{-g})\varepsilon_{\alpha\beta\gamma\delta}\Big\} \end{aligned} \tag{9}$$

and

$$-\frac{1}{32\pi G}\int_{\partial\mathcal{U}} \omega^{\alpha\beta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} \tag{10}$$

Note there are 2 exterior derivatives in the bulk term. f is a 1-form so df is a 2-form and can be assumed to be $df = Cf \wedge f$ with C a(n) (indexed) scalar function. $\sqrt{-g} = \sqrt{|\det g_{\alpha\beta}|}$ is a scalar function so $d\sqrt{|\det g_{\alpha\beta}|}$ is a 1-form. Now for the metric $g = g_{\alpha\beta}$ (not $g = \det g_{\alpha\beta}$) $0 = Dg = dg - \omega g - \omega^{T} g$ where D is the exterior covariant derivative, i.e. $dg_{\alpha\beta} = g_{\alpha\gamma}\omega^{\gamma}{}_{\beta} + \omega^{\gamma}{}_{\alpha} g_{\gamma\beta}$. This is the metric condition. It is assumed [7] so that the metric g is preserved under parallel translation of vectors. This is also an assumption made in GR but is only a simplifying assumption of Differential Geometry. It is the assumption that preserves lengths and angles of vectors under parallel translation. The relation $d\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}dg_{\alpha\beta}$ is an ordinary matrix relation. Finally since ω is a 1-form $\omega = Ff$ with F a $\mathbf{gl_4}$ function. Thus these relations imply (9) is free of exterior derivatives except for (possibly) the ones in the integrand measure, the fs.

If there is a differential relation between the bulk and surface terms, i.e. if exterior derivative (the differential) of the integrand of the surface term = the integrand of the bulk term, then

$$d\omega^{\alpha\beta} = \omega^{\alpha}_{\ \zeta} \wedge \omega^{\zeta\beta} \tag{11}$$

This is because taking the exterior derivative of the surface integrand Eqn. (10) just undoes the steps in Eqn. (7) and cancels the terms in Eqn (9).

Then $R = d\omega + \omega \wedge \omega = 2\omega \wedge \omega$. This imposes a condition on R. Perhaps more importantly it defines $d\omega$. If $R \neq 2\omega \wedge \omega$ then there is no differential relation between the bulk and surface terms. This is true irregardless of whether assumptions are made for df or $d\sqrt{-g}$. These results follow only from the G&S gauge formulation of GR. Again if there is such a differential bulk-surface relation then the bulk and surface terms contain the same information, one being the differential of the other.

1.3 Padmanabhan's differential bulk-surface relation

Now it is desired to reduce the above to GR to compare with Padmanabhan. It has already been shown above that $S_{GS}[\beta, \omega]$ reduces to $S_{EH}[dx, \Gamma]$ and yield the EE.

Reiterating, for the metric $g_{\alpha\beta}$ to be an isometry [7] $0 = Dg = dg - g\omega - \omega^T g$. A connection ω which satisfies this condition is called metric and was assumed by Einstein in GR. In coördinates $dg_{\alpha\beta} = g_{\alpha\gamma}\omega^{\gamma}_{\ \beta} + \omega^{\gamma}_{\ \alpha} g_{\gamma\beta} = \omega_{\alpha\beta} + \omega_{\beta\alpha}$. Also $d\sqrt{-g} = \dfrac{1}{2}\sqrt{-g} g^{\alpha\beta} dg_{\alpha\beta}$. So $d\sqrt{-g} = \omega^{\zeta}_{\ \zeta}\sqrt{-g}$. Further in GR $0 = T = Df = df + \omega \wedge f$, also assumed by Einstein. Thus $df^{\alpha} = -\omega^{\alpha}_{\ \beta} \wedge f^{\beta}$.

Substituting for $d\sqrt{-g}$ and df from the previous 2 lines into (8) gives

$$\begin{aligned} S_{GS}[\beta, \omega] \; = \; & -\frac{1}{32\pi G}\left[\int_{\mathcal{U}} \omega^{\alpha}_{\ \zeta} \wedge \omega^{\zeta\beta} \wedge f^{\gamma} \wedge f^{\delta} \sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} + 2\int_{\mathcal{U}} \omega^{\alpha\beta} \wedge f^{\gamma} \wedge \omega^{\delta}_{\ \zeta} \wedge f^{\zeta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}\right. \\ & \left. + \omega^{\alpha\beta} \wedge f^{\gamma} \wedge f^{\delta} \wedge \omega^{\zeta}_{\ \zeta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} + \int_{\mathcal{U}} d(\omega^{\alpha\beta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta})\right] \end{aligned} \tag{12}$$

Applying Stokes Theorem and setting the 1-form $\omega^{\alpha}_{\ \gamma} = \omega^{\alpha}_{\ \gamma\zeta} f^{\zeta}$ gives

$$\begin{aligned} S_{GS}[\beta, \omega] \; = \; & -\frac{1}{32\pi G}\left[\int_{\mathcal{U}} \omega^{\alpha}_{\ \zeta\eta}\omega^{\zeta\beta}_{\ \ \ \kappa} f^{\eta} \wedge f^{\kappa} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}\right. \\ & -2\int_{\mathcal{U}} \omega^{\alpha\beta}_{\ \ \ \eta}\omega^{\delta}_{\ \zeta\kappa} f^{\eta} \wedge f^{\kappa} \wedge f^{\gamma} \wedge f^{\zeta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} \\ & + \omega^{\alpha\beta}_{\ \ \ \eta}\omega^{\zeta}_{\ \zeta\kappa} f^{\eta} \wedge f^{\kappa} \wedge f^{\gamma} \wedge f^{\delta}\varepsilon_{\alpha\beta\gamma\delta}\sqrt{-g} \\ & \left. + \int_{\partial\mathcal{U}} \omega^{\alpha\beta}_{\ \ \ \zeta} f^{\zeta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}\right] \end{aligned} \tag{13}$$

This expression embodies the assumptions of GR except the integrands are not contracted to scalar functions like in GR. (Recall $R = d\omega + \omega \wedge \omega$, $R^{ab} = \frac{1}{2}R^{\alpha\beta}_{\ \ \ \gamma\delta} f^r \wedge f^s$, and $R^{\alpha\beta}_{\ \ \ \gamma\delta} \to R^{\alpha\beta}_{\ \ \ \alpha\beta}$.) Also the basis of the cotangent space in GR should be holonomic, $f \to dx$.

These integrands will reduce under contraction to Padmanabhan's bulk and surface Lagrangians. The expressions of Padmanabhan's bulk and surface terms (neglecting the integrand measures) are [1]

$$\text{bulk: } \sqrt{-g}g^{ik}\left(\Gamma^{m}_{\ il}\Gamma^{l}_{\ km} - \Gamma^{l}_{\ ik}\Gamma^{m}_{\ lm}\right) \quad \text{surface: } -\sqrt{-g}\left(g^{ck}\Gamma^{m}_{\ km} - g^{ik}\Gamma^{c}_{\ ik}\right) \tag{14}$$

Padmanabhan's latin indices which run $c, i, k, l, m = 0, \ldots, 3$ have been retained.

Getting the terms to match requires some trial and error. For the first term on the rhs of Eqn. (13), $\eta \to \beta$ and $\kappa \to \alpha$. Then $\omega^{\alpha}_{\ \zeta\eta}\omega^{\zeta\beta}_{\ \ \kappa} \xrightarrow[\kappa\to\alpha]{\eta\to\beta} \omega^{\alpha}_{\ \zeta\beta}\omega^{\zeta\beta}_{\ \ \alpha} = \omega^{\zeta\beta}_{\ \ \alpha}\omega^{\alpha}_{\ \zeta\beta} \to g^{ik}\omega^{m}_{\ il}\omega^{l}_{\ mk}$ is the same as Padmanabhan's first bulk term in (14), if $\omega^{\alpha}_{\ \zeta\eta}$ is Γ^{i}_{jk}. Also it was necessary to assume ω is symmetric in its lower indices as is the Christoffel symbol Γ. The contractions make the integrand measure or volume form be $-f^{\beta} \wedge f^{\alpha} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}$. This agees with the conventional EH action as in Padmanabhan as the outside sign is taken to be $+$ and not $-$ as here in G&S. The other terms are done almost the same. For the second term on the rhs of (13) $\omega^{\alpha\beta}_{\ \ \eta}\omega^{\delta}_{\ \zeta\kappa} \xrightarrow[\kappa\to\alpha,\,\zeta\to\delta]{\eta\to\beta} \omega^{\alpha\beta}_{\ \ \beta} \wedge \omega^{\delta}_{\ \delta\alpha} \to g^{ik}\omega^{l}_{\ ik}\omega^{\delta}_{\ \delta\alpha}$. In the third term on the rhs of (13) again as before $\eta \to \beta$, $\kappa \to \alpha$, and $\zeta \to \delta$. The same term results as in the previous case with -2 of the previous and $+1$ of the last giving the second bulk term. The volume forms all become $-f^{\beta} \wedge f^{\alpha} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}$.

In the boundary term in (13) $\omega^{\alpha\beta}_{\ \ \zeta}$ must be contracted over both α and β. Writing this out

$$\begin{aligned}
&\int_{\partial\mathcal{U}} \omega^{\alpha\beta}_{\ \ \zeta} f^{\zeta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta} \\
= &\int_{\partial\mathcal{U}} \omega^{\alpha\zeta}_{\ \ \zeta} f^{\zeta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\zeta\gamma\delta} + \int_{\partial\mathcal{U}} \omega^{\zeta\beta}_{\ \ \zeta} f^{\zeta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\zeta\beta\gamma\delta} \\
= &\int_{\partial\mathcal{U}} \left(\omega^{\alpha\zeta}_{\ \ \zeta} - \omega^{i\alpha}_{\ \ i}\right) f^{\zeta} \wedge f^{\gamma} \wedge f^{\delta}\sqrt{-g}\varepsilon_{\alpha\zeta\gamma\delta}
\end{aligned} \tag{15}$$

which is the surface term when account is taken of the overall $-$ in Eqn. (13).

The following diagram summarizes the results.[1]

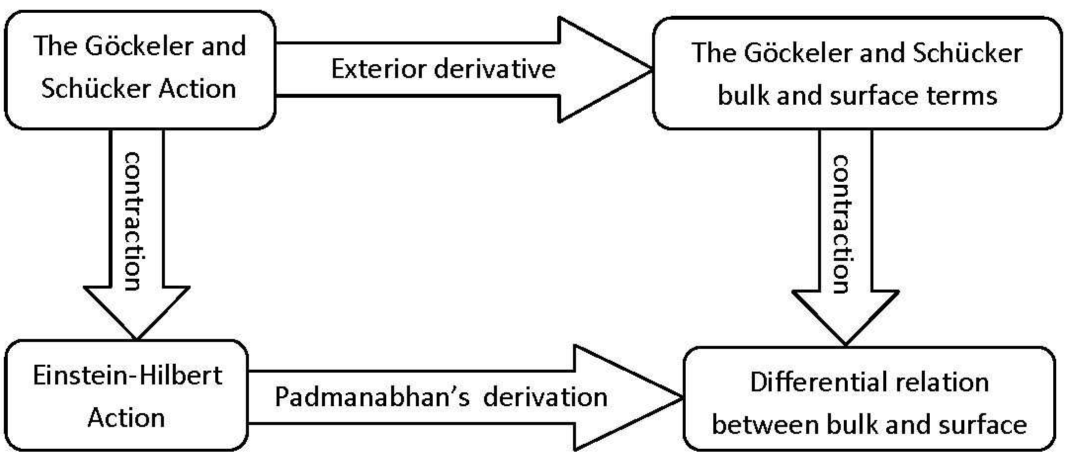

Figure 1: Commutative diagram summarizing above derivation.

1.4 Implications of the results

The above calculations imply $d\omega = \omega \wedge \omega$ so that $R = 2\omega \wedge \omega$ or if preferred $R = 2d\omega$. What may be more significant however is that $d\omega = \omega \wedge \omega$ defines $d\omega$.

To try to understand such results recall GR requires the torsion $T = 0$. Then $0 = T = df + \omega \wedge f$ implies $df = -\omega \wedge f$. So $-\omega \wedge f = df = \frac{1}{2}Cf \wedge f$, where the assumption $df^{\alpha} = \frac{1}{2}C^{\alpha}_{\ \beta\gamma}f^{\beta} \wedge f^{\gamma}$ has been made because

[1] This diagram is due to Louis Kauffman.

df is a 2-form, with $C^{\alpha}_{\beta\gamma}$ antisymmetric in β and γ. Then $\left(\frac{1}{2}Cf - \omega\right) \wedge f = 0$, and by tensor properties of $\wedge$, $\omega = \frac{1}{2}Cf$. Thus $R = 2d\omega = Cdf$.

Conversely suppose there is a differential relation between the bulk and surface terms so that $R = 2\omega \wedge \omega$ but the torsion is present and arbitrary. This is outside the scope of GR. Then $\omega \wedge T = \omega \wedge df + \omega \wedge \omega \wedge f = \omega \wedge df + \frac{1}{2} R \wedge f$. This is a constraint on the torsion T. If $T = 0$ as in GR then $R \wedge f = -2\omega \wedge df$ is a another condition on R. Given the limitations of time it has not been possible to develop this further or to represent these conclusions in GR.

2 The entropy functional of Padmanabhan and Paranjape

In [8] Padmanabhan and Paranjape postulate an entropy functional based on elasticity. Gravity as elasticity is Sakharovs paradigm. For coördinates x^a, the elastic deformation vector ξ^a with $x^a \to x^a + \xi^a(x)$, describes the elastic displacement of a solid. In this section latin indices $a, b = 0, \ldots, 3$ will be used throughout. The quadratic elastic functional is applied to spacetime and is postulated by them to be

$$S[\xi^\alpha] = \int_{\mathcal{V}} d^D x \sqrt{-g} \left(4 P_{cd}{}^{ab} \nabla_a \xi^c \nabla_b \xi^d - T_{ab} \xi^a \xi^b\right) \tag{16}$$

In elasticity the first term on the rhs would correspond to the quadratic displacement field with $P_{cd}{}^{ab}$ constant. The ξ^a are now being taken as spacetime vectors. The second term corresponds to an energy-matter term and will be ignored. From this action principle Padmanabhan and Paranjape can derive the Einstein equations in 4 and higher dimensions as well as higher order corrections using the Lanczos-Lovelock lagrangians [3]. Further Padmanabhan and Paranjape are able with this action to derive Wald's Entropy [9].

Padmanabhan and Paranjape show the first term in the integrand in (16)

$$\mathcal{I} = 4 P_{cd}{}^{ab} \nabla_a \xi^c \nabla_b \xi^d \boldsymbol{\epsilon} \tag{17}$$

where $\boldsymbol{\epsilon} = \frac{1}{D!} \epsilon_{a_1 \cdots a_D} \boldsymbol{\omega}^{a_1} \wedge \cdots \wedge \boldsymbol{\omega}^{a_D}$, $\boldsymbol{\omega}^i = \mathbf{d}x^i$, $(\nabla_b \xi^a)\, \omega^b = \mathbf{d}\xi^a + \boldsymbol{\omega}^a{}_b \xi^b$, $\boldsymbol{\omega}^a{}_b = \Gamma^a_{bc} \boldsymbol{\omega}^c$ can be written as

$$\mathcal{I} = 4 * \mathbf{P}_{ab} \wedge (\mathbf{d}\boldsymbol{\xi})^a \wedge (\mathbf{d}\boldsymbol{\xi})^b \tag{18}$$

where $\mathbf{P}^{ab} = \frac{1}{2!} P^{ab}{}_{cd} \boldsymbol{\omega}^c \wedge \boldsymbol{\omega}^d$, $\boldsymbol{\xi} = \xi^a \mathbf{e}_a$, $\mathbf{e}_a$ basis vectors, $(\mathbf{d}\boldsymbol{\xi})^a = (\nabla_b \xi^a)\, \boldsymbol{\omega}^b$ and $*\mathbf{P}_{ab} = \frac{1}{(D-2)!} \frac{1}{2!} P_{cd}{}^{ab} \epsilon_{cda_1 \cdots a_{D-2}} \boldsymbol{\omega}^{a_1} \wedge \cdots \wedge \boldsymbol{\omega}^{a_{D-2}}$. (The boldface type is just following Padmanabhan and Paranjape's conventions.) The ϵ here is the permutation symbol. The $\sqrt{-g}$ is kept in the integrand measure and not used in the Hodge star $*$. Thus (ignoring the matter term) $S[\xi]$ becomes

$$S[\xi] = 4 \int_{\mathcal{V}} *\mathbf{P}_{ab} \wedge (\mathbf{d}\boldsymbol{\xi})^a \wedge (\mathbf{d}\boldsymbol{\xi})^b \tag{19}$$

It will now be shown that this Action can be rewritten as a Göckeler and Schücker action $S_{GS}[\beta, \omega]$ of the previous section.

Writing the integrand out in full

$$\begin{aligned} & *\mathbf{P}_{ab} \wedge (\mathbf{d}\boldsymbol{\xi})^a \wedge (\mathbf{d}\boldsymbol{\xi})^b \\ = \; & \frac{1}{(D-2)!} \frac{1}{2!} P_{ab}{}^{cd} \epsilon_{cda_1 \cdots a_{D-2}} \boldsymbol{\omega}^{a_1} \wedge \cdots \wedge \boldsymbol{\omega}^{a_{D-2}} \wedge \boldsymbol{\beta}^a \wedge \boldsymbol{\beta}^b \end{aligned} \tag{20}$$

with $(\mathbf{d}\boldsymbol{\xi})^a$ being a 1-form set $= \boldsymbol{\beta}^a$. Note these $\boldsymbol{\beta}$s are not holonomic.

On the other hand in D dimensions the integrand in the Göckeler and Schücker action $S_{GS}[\beta, \omega]$ is

$$\begin{aligned} & -R^a{}_b \wedge *(\beta^b \wedge \beta_a) = -R_{ab} \wedge *(\beta^b \wedge \beta^a) = R_{ab} \wedge *(\beta^a \wedge \beta^b) \\ = & \frac{1}{(D-2)!}\frac{1}{2!}R_{abcd}\beta^c \wedge \beta^d \epsilon_{aba_1\cdots a_{D-2}}\sqrt{-g}g^{aa}g^{bb}\beta^{a_1} \wedge \cdots \wedge \beta^{a_{D-2}} \end{aligned} \tag{21}$$

where $R_{ab} = \frac{1}{2!}R_{abcd}\beta^c \wedge \beta^d$. The βs are not necessarily holonomic.

Interchanging the indices a, b and c, d in the previous line

$$\begin{aligned} = & \frac{1}{(D-2)!}\frac{1}{2!}R^{cd}{}_{ab}\beta^a \wedge \beta^b \sqrt{-g}\epsilon_{cda_1\cdots a_{D-2}}\beta^{a_1} \wedge \cdots \wedge \beta^{a_{D-2}} \\ = & \frac{1}{(D-2)!}\frac{1}{2!}R^{cd}{}_{ab}\epsilon_{cda_1\cdots a_{D-2}}\beta^{a_1} \wedge \cdots \wedge \beta^{a_{D-2}} \wedge \beta^a \wedge \beta^b \end{aligned} \tag{22}$$

which is the same as Eqn. (20) if $R^{cd}{}_{ab} = P_{ab}{}^{cd}$ and the β^{a_i}s are chosen to match the $\boldsymbol{\omega}$s. The $\sqrt{-g}$ has been absorbed into the ϵ permutation symbol and the name retained. Thus Padmanabhan and Paranjape's elasticity formulation can be reexpressed in terms of the curvature. It may be mentioned that the Padmanabhan and Paranjape's paper [8] is quite elegant.

Note that if this reformulation is accepted it implies that Wald's entropy can also be derived from the G&S formulation. Also it would imply the gauge invariance of G&S GR would apply in some sense to the Wald entropy. Work on this is ongoing.

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